# On the Solution of Problems of Transient Heat Conduction in Layered Media 

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ABSTRACT
The problems of transient heat conduction in multi-layered objects are studied. A solution of a boundary uniform problem with transient boundary conditions of third type is suggested. The Fourier method of variable separation by eigen functions of the problem and Duhamel's integral are taken as a basis of the solution. The suggested solution form is of explicit form and due to recurrent format of writing basic relations can be useful for numerical calculations and analyses of kinetics of transient heating (cooling) of multi-layered objects.

## KEYWORDS

Boundary problem, Fourier heat equation, multi-layered object, transient boundary conditions of third type, explicit recurrent form of solution

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## Introduction

Many important practical problems of calculating temperature fields in multi-layered objects can be considered one-dimensional. This subject is explored by foreign researchers (Korn and Korn 1983; Pichard, 1985; Vaessen, 1987; Vandana \& Banthia, 1989; Cory \& Rosenhause, 1997; Lekner, 1990; Harmut, 1985; Lindell \& Sihvola, 1995) as well as by domestic researchers (Kudinov, 2011; Kudinov, Kartashov \& Kalashnikov, 2005; Kudinov, Averin \& Stefanyuk, 2008; Kudinov \& Kartashov, 2000; Temnikov, Igonin \& Kudinov, 1982; Bateman, 1958; Falkowski, 1978; Kudryashov, Zavizon \& Betsky, 1998; Bogdanovich, 1978; Basantkumar et al., 2006, Burlakov, Kondrashov \& Maltsev, 2004; Kalashnikov et al., 1995; 1996). Previously, the author suggested the analytical solution of a uniform problem of transient heat conduction in multilayered objects under steady-state boundary conditions of third type (Vendin, 1993; 2013; Vendin \& Trubaev, 2013).

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## Results and Discussion

A solution of that task under transient boundary conditions of third type is described below.

In the general case the mathematical statement of a one-dimensional problem of thermal conductivity for multi-layered objects is defined with the following differential equation system:

$$
\begin{equation*}
\frac{\partial T_{i}(r, t)}{\partial t}=a_{i} \nabla^{2} T_{i}(r, t), \quad \mathrm{xi}-1 \leq \mathrm{r} \leq \mathrm{xi}, \quad \mathrm{i}=1,2, \ldots \mathrm{n} \tag{1}
\end{equation*}
$$

where ai,- temperature conductivity coefficients of the ith layer correspondingly; $\mathrm{Ti}(\mathrm{r}, \mathrm{t})$ - temperature field of the ith layer; $\mathrm{x} 0, \mathrm{xn}$ - coordinates of lower and upper geometrical (free) surface of an object correspondingly;

We also assume that the object is isotropic, i.e. thermophysical parameters in each layer are constant and homogeneous throughout the volume it occupies.

We define free-surface boundary conditions $\mathrm{r}=$ xo ,r $=\mathrm{xn}$ as boundary conditions of third type, assuming that boundary conditions of first and second type can be represented as special cases of boundary conditions of third type.

In this case according to (Kartashov, 2001) we get the following:

$$
\left[T_{1}(r, t)+h_{1} \frac{\partial T_{1}(r, t)}{\partial r}\right]_{r=x_{0}}=\varphi_{1}(t),\left[T_{n}(r, t)+h_{2} \frac{\partial T_{n}(r, t)}{\partial r}\right]_{r=x_{n}}=\varphi_{2}(t)
$$

(2)

The boundary conditions of conjugating temperature fields and heat flows at the boundaries of layer division are generally defined using the following expressions:

$$
\begin{gathered}
\left|T_{i}(r, t)=T_{i+1}(r, t)\right| r=x_{i},\left|\lambda_{i} \frac{\partial T_{i}(r, t)}{\partial r}=\lambda_{i+1} \frac{\partial T_{i+1}(r, t)}{\partial r}\right|_{r=x_{i}} \\
(3) \\
\mathrm{i}=1,2, \ldots \mathrm{n}-1,
\end{gathered}
$$

where $\lambda i$ - thermal conductivity of the ith layer.
The initial distribution of temperature fields in each layer is as follows:

$$
\begin{equation*}
T_{i}(r, 0)=f_{i}(r), \mathrm{i}=1,2, \ldots \mathrm{n} \tag{4}
\end{equation*}
$$

If we present the desired problem solution as a sum of

$$
\begin{equation*}
T_{i}(r, t)=f_{i}(r)+v_{i}(r, t), \tag{5}
\end{equation*}
$$

the problem comes to determining the functions $v_{i}(r, t)$, which are solution of the problem with zero initial conditions and satisfy the following equations:

$$
\begin{align*}
& \frac{\partial v_{i}(r, t)}{\partial t}=a_{i} \nabla^{2} v_{i}(r, t), \quad \mathrm{xi}-1 \leq \mathrm{r} \leq \mathrm{xi}, \quad \mathrm{i}=1,2, \ldots \mathrm{n},  \tag{6}\\
& {\left[v_{1}(r, t)+h_{1} \frac{\partial v_{1}(r, t)}{\partial r}\right]_{r=x_{0}}=\varphi_{1}(t),}
\end{align*}
$$

$$
\begin{gather*}
{\left[v_{n}(r, t)+h_{2} \frac{\partial v_{n}(r, t)}{\partial r}\right]_{r=x_{n}}=\varphi_{2}(t)}  \tag{7}\\
\left|v_{i}(r, t)=v_{i+1}(r, t)\right|_{r=x_{i}},\left|\lambda_{i} \frac{\partial v_{i}(r, t)}{\partial r}=\lambda_{i+1} \frac{\partial v_{i+1}(r, t)}{\partial r}\right|_{r=x_{i}}
\end{gather*}
$$

$$
\begin{equation*}
\mathrm{i}=1,2, \ldots \mathrm{n}-1, v_{i}(r, 0)=0, \mathrm{i}=1,2, \ldots \mathrm{n}, \tag{9}
\end{equation*}
$$

In the general case a problem with time-dependant inhomogeneous boundary conditions can be solved by Duhamel's integral (Kartashov, 2001; Korn \& Korn, 1983):

$$
\begin{equation*}
v_{i}(r, t)=\int_{0}^{t} \frac{\partial}{\partial t} \dot{v}_{i}(r, \tau, t-\tau) d \tau, \quad \text { when } \mathrm{t}>0 \tag{10}
\end{equation*}
$$

where $\dot{v}_{i}(r, \tau, t)$ - problem solution provided that $\tau$ is a parameter.
Then functions $\dot{v}_{i}(r, \tau, t)$ should satisfy the differential equation (5) with initial conditions $\dot{v}_{i}(r, \tau, 0)=0$ and free-surface boundary conditions $\mathrm{r}=\mathrm{x} 0$, xn , as well as conjugating conditions:

$$
\begin{gather*}
{\left[\dot{v}_{1}(r, \tau, t)+h_{1} \frac{\partial \dot{v}_{1}(r, \tau, t)}{\partial r}\right]_{r=x_{0}}=\varphi_{1}(t)} \\
{\left[\dot{v}_{n}(r, \tau, t)+h_{2} \frac{\partial \dot{v}_{n}(r, \tau, t)}{\partial r}\right]_{r=x_{n}}=\varphi_{2}(t)}  \tag{11}\\
\left|\dot{v}_{i}(r, \tau, t)=\dot{v}_{i+1}(r, \tau, t)\right|_{r=x_{i}}, \\
\left|\lambda_{i} \frac{\partial \dot{v}_{i}(r, \tau, t)}{\partial r}=\lambda_{i+1} \frac{\partial \dot{v}_{i+1}(r, \tau, t)}{\partial r}\right|_{r=x_{i}}  \tag{12}\\
\mathrm{i}=1,2, \ldots \mathrm{n}-1
\end{gather*}
$$

Based on the found solution, the functions $\dot{v}_{i}(r, \tau, t)$ are determined using the following expressions:

$$
\begin{equation*}
\dot{v}_{i}(r, \tau, t)=\psi_{i}(r, t)+\sum_{m=0}^{\infty} C_{m}(\tau) \dot{F}_{i, m}\left(\mu_{i, m} r\right) \exp \left(-\mu_{\left.i, m a_{i} t\right)}^{2},\right. \tag{13}
\end{equation*}
$$

And the functions $v_{i}(r, t)$ are the following:

$$
\begin{equation*}
v_{i}(r, t)=\sum_{m=0}^{\infty}\left[-\mu_{i, m}^{2} a_{i} \int_{0}^{t} C_{m}(\tau) \exp \left(\mu_{i, m}^{2} a_{i} \tau\right) d \tau\right] \dot{F}_{i, m}\left(\mu_{i, m} r\right) \exp \left(-\mu_{i, m}^{2} a_{i} t\right) . \tag{14}
\end{equation*}
$$

Where $\dot{F}_{i, m}\left(\mu_{i, m} r\right)$ - problem eigen functions

$$
\begin{equation*}
\dot{F}_{i, m}\left(\mu_{i, m} r\right)=\left[\prod_{k=1}^{i} Z_{k}\right] \times\left[Y_{1}\left(\mu_{i, m} r\right)+B_{i, m} Y_{2}\left(\mu_{i, m} r\right)\right], \mathrm{i}=1,2, \ldots \mathrm{n} \tag{15}
\end{equation*}
$$

$$
\begin{gather*}
B_{1, m}=-\frac{Y_{1}\left(\mu_{1, m} x_{0}\right)+h_{1} Y_{1}^{\prime}\left(\mu_{1, m} x_{0}\right)}{Y_{2}\left(\mu_{1, m} x_{0}\right)+h_{1} Y_{2}^{\prime}\left(\mu_{1, m} x_{0}\right)}  \tag{16}\\
B_{i, m}=\frac{\lambda_{i} \frac{Y_{1}^{\prime}\left(\mu_{i, m} x_{i-1}\right)}{Y_{1}\left(\mu_{i, m} x_{i-1}\right)}-\lambda_{i-1} \frac{Y_{1}^{\prime}\left(\mu_{i-1, m} x_{i-1}\right)+B_{i-1} Y_{2}^{\prime}\left(\mu_{i-1, m} x_{i-1}\right)}{Y_{1}\left(\mu_{i-1, m} x_{i-1}\right)+B_{i-1} Y_{2}\left(\mu_{i-1, m} x_{i-1}\right)}}{\lambda_{i}^{\prime} \frac{Y_{2}^{\prime}\left(\mu_{i, m} x_{i-1}\right)}{Y_{2}\left(\mu_{i, m} x_{i-1}\right)}-\lambda_{i-1} \frac{Y_{1}^{\prime}\left(\mu_{i-1}, m x_{i-1}\right)+B_{i-1} Y_{2}^{\prime}\left(\mu_{i-1, m} x_{i-1}\right)}{Y_{1}\left(\mu_{i-1, m} x_{i-1}\right)+B_{i-1} Y_{2}\left(\mu_{i-1, m} x_{i-1}\right)}} \times \frac{Y_{1}\left(\mu_{i, m} x_{i-1}\right)}{Y_{2}\left(\mu_{i, m} x_{i-1}\right)} \\
\mathrm{i}=2,3, \ldots \mathrm{n} . \tag{17}
\end{gather*}
$$ determined according to the equation

$$
\begin{gather*}
Y_{1}\left(\mu_{n, m} x_{n}\right)+h_{2} Y_{1}^{\prime}\left(\mu_{n, m} x_{n}\right)+B_{n, m} Y_{2}\left(\mu_{n, m} x_{n}\right)+h_{2} Y_{2}^{\prime}\left(\mu_{n, m} x_{n}\right)=0, \\
\mathrm{~m}=0,1,2, \ldots  \tag{19}\\
C_{m}(\tau)=-\left[\sum_{i=1}^{n} \lambda_{a_{i}} \int_{i}^{x_{i}} x_{i-1}(r, \tau) G(r) \dot{F}_{i, m}\left(\mu_{i, m} r\right) d r\right], \sum_{i=1}^{n} \dot{J}_{i}^{2},  \tag{20}\\
\psi_{i}(r, \tau)=\varphi_{1}(\tau)+\left[\dot{\alpha}_{i}(\tau) \varphi_{2}(\tau)-\varphi_{1}(\tau)\right] \times\left[\xi(r)+\dot{\beta}_{i}(\tau)\right],  \tag{21}\\
\dot{\beta}_{1}(\tau)=-\left[\xi\left(x_{0}\right)+h_{1} \xi^{\prime}\left(x_{0}\right)\right]=0, \dot{\beta}_{i}(\tau)=\frac{\lambda_{i}}{\lambda_{i-1}}\left[\xi\left(x_{i-1}\right)+\dot{\beta}_{i-1}\right]-\xi\left(x_{i-1}\right), \mathrm{i}= \\
\dot{\alpha}_{i}=\frac{\lambda_{n}}{\lambda_{i+1}} \times \frac{1}{\xi\left(x_{n}\right)+\dot{\beta}_{n}(\tau)+h_{2} \xi^{\prime}\left(x_{n}\right)} \quad, \mathrm{i}=1,2, \ldots \mathrm{n} .  \tag{22}\\
\dot{J}_{i}^{2}=\frac{\lambda_{i}}{a_{i}} \int_{i-1}^{x_{i}} G(r) \dot{F}_{i, m}^{2}\left(\mu_{i, m}^{r}\right) d r \tag{23}
\end{gather*}
$$

Weight function $G(r)$, as well as specific type of the functions $\xi(r)$ and $\dot{F}_{i}\left(\mu_{i} r\right)$ are found using the following expressions:
a) Cartesian coordinate system:

$$
\begin{equation*}
G(r)=1, \xi(r)=r, Y_{1}\left(\mu_{i} r\right)=\sin \left(\mu_{i} r\right), Y_{2}\left(\mu_{i} r\right)=\cos \left(\mu_{i} r\right) . \tag{24}
\end{equation*}
$$

b) spherical coordinate system:

$$
\begin{equation*}
G(r)=r^{2}, \xi(r)=\frac{1}{r}, Y_{1}\left(\mu_{i} r\right)=\frac{1}{r} \sin \left(\mu_{i} r\right), Y_{2}\left(\mu_{i} r\right)=\frac{1}{r} \cos \left(\mu_{i} r\right) . \tag{25}
\end{equation*}
$$

c) cylindrical coordinate system:

$$
\begin{equation*}
G(r)=r, \xi(r)=\ln r, Y_{1}\left(\mu_{i} r\right)=J_{0}\left(\mu_{i} r\right), Y_{2}\left(\mu_{i} r\right)=N_{0}\left(\mu_{i} r\right) . \tag{26}
\end{equation*}
$$

## Important note:

Sometimes when solving problems for a solid sphere or cylinder the found solution requires limitation at the sphere center or at the cylinder axis. Then lower and upper boundary conditions are as follows:

$$
\begin{equation*}
\left[\frac{\partial T_{1}(r, t)}{\partial r}\right]_{r=x_{0}}=0,\left[T_{n}(r, t)+h_{2} \frac{\partial T_{n}(r, t)}{\partial r}\right]_{r=x_{n}}=\varphi_{2}(t) \tag{2.27}
\end{equation*}
$$

In this case we should assume the following in the found solution for multilayered objects:

$$
\begin{equation*}
B_{1, m}=0, \psi_{i}(r, \tau)=\varphi_{2}(\tau), \mathrm{i}=1,2 \ldots \mathrm{n} \tag{2.28}
\end{equation*}
$$

and all further calculations are made in accordance with the main solution.

## Conclusion

Thus, we have got the general solution of a boundary uniform problem with transient boundary conditions of third type. The suggested solution form is of explicit form and due to recurrent format of writing basic relations can be useful for numerical calculations and analyses of kinetics of transient heating (cooling) of multi-layered objects.

Different partial solutions of such problems can be directly written with regard to boundary conditions (2), as well as expressions (4), (5), (14) and (2.24) - (2.28).

## Disclosure statement

No potential conflict of interest was reported by the authors.

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