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Investigation and Evaluation of Operational Matrix in Order to Solve the Partial Differential Equation

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ABSTRACT

We need differential equations for modeling and analyzing a huge amount of issues. Fractional calculus is a branch of mathematical analysis that is used in many fields of mathematical and engineering sciences such as electrical networks, fluid mechanics, control theory, electromagnetism, biology, chemistry, propagation and viscoelasticity. The most important topics in mathematics are differential equations and integral equations which are very practical and have a special place in various sciences, especially engineering sciences. We used approximate methods to obtain the results because we cannot use analytical clustering for this kind of equations. The aim of this paper is to investigate the operational matrix in order to solve the partial differential equation.

Keywords: differential equations, operating matrix, minor derivatives, gamma function, integral equations, special functions, fractional problems

INTRODUCTION

Fractional Calculus gained great interest in the researchers' community due to its wide applications in several branches of Applied Mathematics and Sciences. Many dynamical systems can be described in a more precise way by using fractional order differential equations, due to the nonlocal nature of fractional derivative. Hence in many cases such equations appear as important alternatives for integer order differential equations. Various natural systems in the fields such as viscoelasticity, electrical circuits and nonlinear oscillations of earthquake show an intermediate behavior which can only be modeled using fractional order differential equations. The numerical methods for solving these equations includes Laplace transforms, serial power method, Fourier transforms, Special vector expansion, Adomian split method, Repeat change method, Fractional transformation method, Fractional difference method, Homotopy analysis method, Operational matrix method, Generalized transformation method, Time discretization method and other methods. The purpose of solving a problem of differential equations by partial derivatives is solving an equation that is true in some physical conditions. If the physical condition of the problem is a primary (original), then this problem is called the initial value problem.

If the physical condition of the problem is of a boundary type, the problem is primary and if the problem is boundary type, then it is called the boundary-primary problem. Several fundamental investigations have been made on deficit differential equations and fractional derivatives. These studies are described as introductions or descriptions for the theory of fractional derivatives and differential deficit equation. It is important to find the exact or approximate answer for differential equations but, except for a limited number of these equations, finding an analytic solution is difficult or impossible. Also, so far numbers of numerical methods have been

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considered for solving differential deficit equations, such as Adomian decomposition methods, homotopy decomposition method, and repetitive method. Spectral methods have provided a powerful tool for solving many of the differential equations in the fields of science and engineering. Here, the high precision and ease of use of these methods are two effective features which encourage many researchers to use them in various equations. Certain types of spectral methods which are more practical include Galerkin method, Spatial method, and Taoist methods. A. Saadatmandi (2014) and M. Dehghan, S. Abdi-mazraeh & M. Lakestani (2014) introduced the transmitted Legendre operational matrix for fraction derivatives and used it with methods and spectra for linear and nonlinear deficit differential equations, depend on the initial conditions. Recently, Bahravi et al., used the translated Chebyshev polynomials for multivariate linear differential equations with variable coefficients. Also, Ismaili proposed a computational technique based on the spatial method and the Manse polynomials for solving differential deficit equations. The aim of this paper is the introduction of the matrix of Jacobi polynomials transmitted for fraction derivatives which is based on spectral methods for solving differential equations of linear and nonlinear fractions with initial or boundary conditions.

Preliminaries and Notations

Definition: The Riemann-Liouville deficit integral of α is defined as follows:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * f(x) \qquad x \in \Omega, I^0 f(x) = f(x)$$

where Γ is the Gamma function and $*$ is convolution multiplication. For the I^{α} operator, the following equations are established:

$$I^{\alpha}(x-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-\alpha)^{\alpha+\beta}, \quad \alpha \geq 0, \beta > -1,$$

$$I^{\alpha}\left(I^{\beta} f(x)\right) = I^{\alpha+\beta} f(x) = I^{\beta}\left(I^{\alpha} f(x)\right).$$

Definitions and Introductory Theorems

Definition: Any equation that contains an associated variable and its derivatives relative to one or more independent variables is called a differential equation.

Differential equations are divided into two general categories:

Ordinary differential equations (ODEs)

Partial Differential equations (PDEs)

Each differential equation, which contains a function of two or more variables and derivatives of a function relative to independent variables, is called a partial differential equation.

Definition: The right side and the left Liouville fraction integral ($\alpha \geq 0$) of $u(x)$ are defined by the following equations:

$$I_{+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^x u(\xi)(x-\xi)^{\alpha-1} d\xi\,, \quad \alpha > 0, 0 < x < L,$$

$$I_{-}^{\alpha} u(t) = \frac{1}{\Gamma\,(\alpha)} \int_x^L u(\xi)(\xi-x)^{\alpha-1} d\xi\,, \quad \alpha > 0, 0 < x < L,$$

$$I_{\mp}^{0} u(t) = u(x).$$

The $\mathrm{I}_{\mp}^{\alpha}$ operator is defined as follows:

$$I_{+}^{\alpha} x^{i} = \frac{\Gamma(i+1)}{\Gamma(i+1+\alpha)} x^{i+1},$$

$$I_{-}^{\alpha}(x-L)^{i} = \frac{(-1)^{i}\Gamma(i+1)}{\Gamma(i+1+\alpha)}(L-x)^{i+\alpha}.$$

Definition: The right and left derivatives of the Riemann-Liouville deficit integral of α are defined as follows

$$D_{*+}^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x\frac{u(\xi)}{(x-\xi)^{\alpha-n+1}}d\xi,$$

$$D_{*+}^{\alpha}u(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^L\frac{u(\xi)}{(\xi-x)^{\alpha-n+1}}d\xi.$$

For $n-1<\alpha\leq n$, $n\in N$, n is the smallest integer number that is larger than α.

Definition: The derivatives of the right and left of Caputo-Dzherbashyn function of α are as follows:

$$D_{+}^{\alpha}u(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x\frac{u^{(n)}(\xi)}{(x-\xi)^{\alpha-n+1}}d\xi,$$

$$D_{-}^{\alpha}u(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^L\frac{u^{(n)}(\xi)}{(\xi-x)^{\alpha-n+1}}d\xi.$$

The $D_{\mp}^{\alpha}$ operator is true in following equation:

$$D_{\mp}^{\alpha}J_{\mp}^{\alpha}u(x)=u(x),$$

$$D_{\mp}^{\alpha}J_{\mp}^{\alpha}u(x)=u(x)-\sum_{i=0}^{[\alpha]-1}u^{(i)}(O^{+})\frac{x^i}{i!},$$

$$D_{+}^{\alpha}x^i=\begin{cases}0, & \text{for } i\in N_0 \text{ and } i<\lceil\alpha\rceil\\ \dfrac{\Gamma(i+1)}{\Gamma(i+1-\alpha)}x^{i-\alpha}, & \text{for } i\in N_0 \text{ and } i\geq\lceil\alpha\rceil\end{cases}$$

$$D_{\mp}^{\alpha}\big(\varepsilon\psi(x)+\epsilon\phi(x)\big)=\varepsilon D_{\mp}^{\alpha}\phi(x)\psi(x)+\epsilon D_{\mp}^{\alpha}\,\phi(x).$$

ε and ϵ are the constants which have been defined as follow:

$$D_{-}^{\alpha}(x-L)^i=\begin{cases}0, & \text{for } i\in N_0 \text{ and } i<\lceil\alpha\rceil\\ \dfrac{(-1)^i\Gamma(i+1)}{\Gamma(i+1-\alpha)}(L-x)^{i-\alpha}, & \text{for } i\in N_0 \text{ and } i\geq\lceil\alpha\rceil\end{cases}$$

where, $\lceil * \rceil$ is the ceiling function and $N_0=\{0,1,2,\dots\}$.

Definition: The Riesz fractional derivatives of order α of $u(x)$ is defined as

$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x,t)=-C_{\alpha}\left(D_{*}^{\alpha}+u(x,t)+D_{*}^{\alpha}-u(x,t)\right)$$

where,

$$C_{\alpha}=\frac{1}{2\cos\frac{\pi\alpha}{2}},\alpha\neq 1.$$

Definition: assume that $B(D_E)$ is $\partial\partial\partial$ functions of rank F, then in D_E:

$$\int_{\psi(t+l)}|F(z)dz|\to 0,\quad t\to\pm\infty$$

$$L=\left\{iv:|v|<d\leq\frac{\pi}{2}\right\},$$

Since D_E is bounded, then:

$$N(F)=\int_{\partial D_E}|F(z)dz|<\infty$$

Definition: a m-function of block-pulse is defined as follow:

$$\phi_i(t)=\begin{cases}1, & \dfrac{iT}{m}\leq t<\dfrac{(i+1)T}{m}\\ 0, & \text{otherwise.}\end{cases}$$

where, $i=0,1,\dots,m-1$, with a positive integer value for m, $T=1$ and BPFs for each $[0,1)$ is equal $h=1/m$.

According to the definition of BPFs:

$$\phi_i(t)\phi_j(t) = f(x) = \begin{cases} \phi_i(t), & i = j \\ 0, & i \neq j. \end{cases}$$

and $i, j = 0,1, \dots, m-1$.

The other property is orthogonality. It is clear that

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h\,\delta_{ij},$$

where δ_{ij} is the Kronecker delta.

The third property is completeness. For every $f \in \mathcal{L}^2([0,1))$, when m approaches to the infinity, Parseval's identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 ||\phi_i(t)||^2,$$

$$f_i = \frac{1}{h}\int_0^1 f(t)\phi_i(t)dt.$$

Theorem: let $\phi_{L,M}(t)$ be the shifted Jacobi vector defined in Eq.

Then the left-sided Caputo fractional derivative of order $v > 0$ of $\phi_{L,M}(t)$ can be expressed as

$$D_+^v \phi_{L,M}(x) \cong D_+^{(v)} \phi_{L,M}(x)$$

where $D_+^{(v)}$ is the $(M+1)\times(M+1)$ Jacobi operational of the left sided-fractional derivatives of order v in the Caputo sense, and it is defined as follows:

$$D_+^{(v)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_{v+}(\lceil v\rceil, 0) & S_{v+}(\lceil v\rceil, 1) & S_{v+}(\lceil v\rceil, 2) & \cdots & S_{v+}(\lceil v\rceil, M) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_{v+}(i, 0) & S_{v+}(i, 1) & S_{v+}(i, 2) & \cdots & S_{v+}(i, M) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_{v+}(M, 0) & S_{v+}(M, 1) & S_{v+}(M, 2) & \cdots & S_{v+}(M, M) \end{bmatrix}$$

$$\begin{aligned} &S_{v+}(i,j) \\ &= \sum_{k=\lceil v\rceil}^{i} \frac{(-1)^{i-k}\Gamma(i+\vartheta+1)\Gamma(i+k+\theta+\vartheta+1)\Gamma(\theta+1)L^{-v}}{\Gamma(k+\vartheta+1)\Gamma(i+\theta+\vartheta+1)(i-k)!\,\Gamma(k-v+1)} \\ &\times \sum_{s=0}^{j} \frac{(-1)^{i-s}\Gamma(j+s+\theta+\vartheta+1)\Gamma(s+k+\vartheta-v+1)(2j+\theta+\vartheta+1)j!}{\Gamma(j+\theta+1)\Gamma(s+\vartheta+1)(j-s)!\,s!\,\Gamma(s+k+\theta+\vartheta-v+2)} \end{aligned}$$

Note that in $D_+^{(v)}$, the first $\lceil v\rceil$ rows, are all zeros.

Lemma: the left and right-sided Caputo fractional derivatives are defined via the Riemann-Liouville fractional:

$$D_+^{\alpha}u(x) = D_{*+}^{\alpha}u(x) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{u^{(i)}(0)}{\Gamma(i+1-\alpha)} x^{i-\alpha},$$

$$D_-^{\alpha}u(x) = D_{*-}^{\alpha}u(x) - \sum_{i=0}^{\lceil v\rceil-1} \frac{(-1)^i u^{(i)}(L)}{\Gamma(i+1-\alpha)} (L-x)^{i-\alpha},$$

Therefore, if function $u(x)$ satisfies $u^{(k)}(0) = 0$, $k = 0,1, \dots, \lceil v\rceil - 1$ then $\mathrm{D}_+^{\alpha}u(x)$ and $\mathrm{D}_{*+}^{\alpha}u(x)$ are equivalent.

Theorem: If $\phi' F \in B(D_E)$ then for all $x \in \Gamma$

$$\left| F(x) - \sum_{k=-\infty}^{\infty} F(x_k)S(k,h)o\phi\,(x) \right| \leq \frac{N(F\phi')}{2\pi d \sinh\left(\frac{\pi d}{h}\right)} \leq \frac{2N(F\phi')}{\pi d} e^{-\pi d/h}$$

Moreover, if $F(x) \leq C_{e^{-\propto|\phi(x)|}}$, $x \in \Gamma$, for some positive constants C and α, and if the selection $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$, then

$$\left|F(x) - \sum_{k=-N}^{N} F(x_k) S(k,h) o\phi\,(x)\right| \leq C_2\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right), \quad x \in \Gamma,$$

where C_2 depends on F, d and α.

Theorem: assume that, f, f_m are the exact and approximate solution of equation (1), respectively. Also, suppose that the function $W(f)$ holds in the Liepschitz condition which means for $L \geq 0$:

$$\left|W\left(f_1(x)\right) - W\left(f_2(x)\right)\right| \leq L|f_1(x) - f_2(x)|, x \in \Omega$$

Also, assume that

$$\|a_k\|_\infty \leq N_k, k = 0,1,\dots,r, \text{ and}$$

$$\sum_{k=1}^{r} \frac{N_k T^{\alpha-\beta_k}}{\Gamma(\alpha-\beta_k+1)} + \frac{LN_0 T^\alpha}{\Gamma(\alpha+1)} < 1.$$

$$E_m = \|f - f_m\|_\infty \leq \frac{MT^\alpha h^3}{9\sqrt{3}\,\Gamma(\alpha+1)\left(1 - \sum_{k-1}^{r} \frac{N_k T^{\alpha-\beta_k}}{\alpha-\beta_k+1} + \frac{LN_0 T^\alpha}{\Gamma(\alpha+1)}\right)}$$

where, M is the upper bound of $\left\|v^{(3)}\right\|_\infty$.

METHODOLOGY

Assume the following function

$$u(x) = y(x) + (a-b)x - a.$$

Therefore we consider the following Bagley-Torvik equation

$$A_1 u^{(2)} + A_2 u^{\left(\frac{3}{2}\right)} + A_3 u = g(x), \qquad x \in [0,1],$$

So, by taking account the boundedness condition:

$$u(0) = 0, \quad u(1) = 0,$$

and

$$g(x) = f(x) + A_3\left((a-b)x - a\right)$$

Now, we approximate solution for $u(x)$, in Eq

$$u(x) = u_M(x) = \sum\nolimits_{k=-N}^{N} ukS_k(x),$$

where, $uk = u(x_k)$ and $M = 2N+1$. It is worth pointing out that $u_M(x) = 0$ when x tends to 0 or 1.

Therefore

$$\frac{d}{dx}[S(k,h)o\phi(x)] = \phi'(x)\frac{d}{d\phi}[S(k,h)o\phi(x)]$$

Thus, using aforementioned Equation we get

$$\frac{d}{dx}S_k\Big|_{x=x,} = \frac{1}{h}\phi'(x_j)\delta_{kj}^{(1)}.$$

Similarly, we will calculate the second derivative:

$$\frac{d}{dx}S_k(x)\Big|_{x=x,} = \frac{1}{h}\phi''(x_j)\delta_{kj}^{(1)} + \frac{1}{h^2}[\phi'(x_j)]^2\delta_{kj}^{(2)}$$

The second-order derivative x_j is obtained from the following relationships:

$$u'_M(x_j) = \sum_{k=-N}^{N} uk\left\{\frac{1}{h}\phi'(x_j)\delta_{kj}^{(1)}\right\},$$

$$u''_M(x_j) = \sum_{k=-N}^{N} uk\left\{\frac{1}{h}\phi''(x_j)\delta_{kj}^{(1)} + \frac{1}{h^2}[\phi'(x_j)]^2\delta_{kj}^{(2)}\right\}.$$

Operational Matrix

To calculate the operator $\int_0^t \varphi_i(\tau)$, we use the following equation:

$$\int_0^t \varphi_i(\tau)\,dt = \begin{cases} 0, & t < ih \\ t - ih, & ih \le t < (i+1)h \\ h, & (i+1)h \le t < 1. \end{cases}$$

The variable $t - ih$ is equal to $h/2$ in $[ih, (i+1)]$. The approximation of $t - ih$ for $ih \le t \le (i+1)$ is equal to $h/2$. The operator $\int_0^t \varphi_i(\tau)$ is obtained from BPFs:

$$\int_0^t \varphi_i(\tau)\,dt \simeq \left[0 \quad \dots \quad 0 \quad \frac{h}{2} \quad h \quad \cdots \quad h\right]\phi t,$$

The component ith in $h/2$ is equal to:

$$\int_0^t \phi_i(\tau)\,dt \simeq P\phi(t)$$

$P_{m\times m}$ is an Operational Matrix as follows:

$$P = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Block-pulse Function

Definition: an m-set of BPFs is defined over a real interval $[0, H)$ as

$$\phi_i(t) = \begin{cases} 1, & \frac{iH}{m} \le t < \frac{(i+1)H}{m}, \\ 0, & Otherwise. \end{cases}$$

$i = 0,1,\dots,m-1$ is a positive integer m. $h = H/m$ and φ_i are BPF. Assume $H = 1$ and BPFs in [0,1) and $h = 1/m$. **Figure 1** indicates the BPFs set of the interval [0,1).

There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness. Let us consider the first m terms of BPFs and write them concisely as an m-vector

$$\phi(t) = [\varphi_0(t) \quad \varphi_1(t) \quad \cdots \quad \varphi_{m-1}(t)]^T, t \in [0,1),$$

where, superscript T indicates transportation. The above representation and disjointness property follows

$$\phi(t)\phi^t(t)V = \tilde{V}\phi(t),$$

where, V is an m-vector and $\tilde{V} = diag(v)$. Moreover, it can be clearly concluded that for any $m \times m$ matrix B

$$\phi^t(t)B\phi(t) = \dot{B}^T\phi(t)$$

where, $\dot{B}$ is an m-vector with elements equal to the diagonal entries of matrix B.

Also

$$\int_0^1 \phi t(t)\,dt = [h \quad h \quad \cdots \quad h]^T = \hbar,$$

and

$$\int_0^1 \phi(t)\phi^T(t)dt = hI,$$

where, I is $m \times m$ identity matrix.

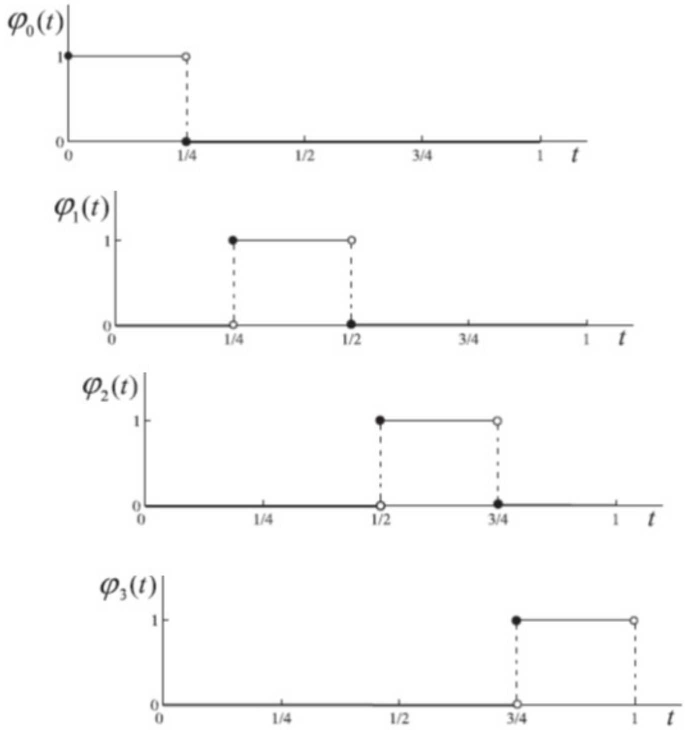

Figure 1. A set of BPFs

BPFs expansion

For function f in [0,1), φ_i, and $i = 0,1,\dots,m-1$, we get:

$$f(t) \simeq \sum_{i=1}^{m-1} f_i \varphi_i(t) = F^T \phi(t) = \phi^T(t) F,$$

where, $F = [f_0 \quad f_1 \quad \cdots \quad f_{m-1}]^T$ and f_i's are defined by

$$f_i = \frac{1}{h}\int_0^1 f(t)\varphi_i(t)\,dt.$$

Nonlinear Integral Equation Solving Method

With regard to approximations of nonlinear integral equation $x(t)$, we get

$$x(t) \simeq C^T T(t),$$

Then we substitute this approximation following equations

$$C^T T(s) = y(s) + \lambda_1 \int_0^s k_1(s,t)$$

$$f\left(t, C^T T(t)\right)dt + \lambda_2 \int_0^1 k_2(s,t)$$

$$g(t, C^T T(t))dt$$

Using the Gaussian integral formula we get

$$\tau_1 = \frac{2}{s_i} t - 1, \qquad \tau_2 = 2t - 1.$$

According to the Chebyshev polynomial in the local points, we get:

$$s_i = cos\left(\frac{i\pi}{N}\right), \qquad i = 0,1,\dots,N,$$

Assume that:

$$H_1(s,t) = K_1(s,t)f\big(t, C^T T(t)\big),$$
$$H_2(s,t) = K_2(s,t)f\big(t, C^T T(t)\big).$$

Therefore, we get

$$s_i = cos\left(\frac{i\pi}{N}\right), \qquad i = 0,1,\dots,N,$$

So,

$$C^T T(s_i) = y(s_i) + \lambda_1 \frac{s_i}{2}\int_{-1}^{1} H_1\left(s_i, \frac{s_i(\tau_1+1)}{2}\right) d\tau_1 + \frac{\lambda_2}{2}\int_{-1}^{1} H_2\left(s_i, \frac{(\tau_2+1)}{2}\right) d\tau_2.$$

Now we can use Clenshaw-Curtis quadrature formula

$$C^T T(s_i) = y(s_i) + \sum_{K=0}^{N}{}'' \omega k \left[\lambda_1 \frac{s_i}{2}\int_{-1}^{1} H_1\left(s_i, \frac{s_i(s_k+1)}{2}\right) + \frac{\lambda_2}{2} H_2\left(s_i, \frac{(s_k+1)}{2}\right)\right],$$

for $i = 0,1,2,\dots,N,$ where

$$\omega k = \frac{4}{N}\sum_{even\ n=0}^{N}{}'' \frac{1}{1-n^2}\cos\left(\frac{nk\pi}{N}\right),$$

DISCUSSIONS AND CONCLUSION

The second Chebyshev wavelets operational matrix of integration and its product operational matrix have been obtained in general and used for solving the integral equations. The present method reduces an integral equation into a set of algebraic equations. Some examples are included to demonstrate the superiority of our method.

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